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Generalized beam-column element on two parameter elastic foundation

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ABSTRACT

A new generalized Bernouli/Timoshenko beam-column element on a two-parameter elastic foundation is presented. This element is based on the exact solution of the differential equation which describes the deflection of the axially loaded beam resting on a two-parameter elastic foundation, and has the ability of optionally taking into consideration shear deformations, semi – rigid connections, and rigid offsets. The equations of equilibrium are formulated for the deformed configuration, in order to take into account the effect of large deflections. Apart from the stiffness matrix, load vectors, for uniform load and non-uniform temperature variation are formulated. The usefulness of the new element in reinforced concrete or steel structures analysis is documented by two examples.

INTRODUCTION

The problem of a beam supported by a flexible medium and subjected not only to transverse but also to axial loading is often encountered in the design of structural members of buildings, aircrafts, ships, machines and other structures. Particularly important are the effects of the axial forces in slender members and structures, in which the effect of the deformations in the structural response cannot be ignored (beam-column effects). To cover this behavior rigorously, the governing equilibrium equations must be formulated with respect to the deformed geometry of the structure; the resulting, geometrically non-linear analysis is referred to as the “second-order analysis” in contradistinction to the ordinary, linear "first order" analysis which neglects the effects of deformations on equilibrium. Second-order analysis is always necessary for the stability consideration of structures. For practical purposes, sophisticated models and sufficiently accurate and simple solutions are required.

The problem of an axially loaded elastic member resting on an elastic foundation has been an important tool for modelling and analysis, especially in the design of building structures, where the superstructure-foundation-soil interaction has to be taken into account. In this area, extensive research has been reported in the literature. In order to model soil behaviour, several approaches have been developed in the past. In the majority of the proposed solutions, the foundation-supporting soil is represented on the basis of the well-known Winkler-hypothesis, which assumes the soil to be made up of continuously distributed not-interconnected discrete springs (Winkler 1867). Thanks to its simplicity, the Winkler model has been extensively used to solve many soil-foundation interaction problems and has given satisfactory results in many practical cases. However, it is a rather crude approximation of the true mechanical behaviour of the ground material. Its discontinuous nature gave rise to the development of various forms of two-parameter elastic foundation models (Filonenko and Borodich 1940; Pasternak 1954; Vlasov and Leontiev 1966), in which the continuity, i.e. the coupling effect between the discrete Winkler springs, is introduced by assuming the Winkler springs to be connected by a shear layer, a membrane or a beam. The two-parameter models describe the soil behaviour more accurately and yet remain simple enough for practical purposes.

On the other hand, most reported solutions for beam-columns on elastic foundations are based on the classical Bernoulli (Bernoulli-Euler or Kirchoff) theory, thus neglecting the effect of transverse shear deformations (e.g., Ting and Mockry 1984). In order to take these deformations into account, analytical solutions for a Timoshenko beam-column resting on an elastic two-parameter foundation have been proposed for the dynamic problem (Wang and Gagnon 1978), and for the first order (linear) analysis (e.g., Shirima and Giger 1992, Onu 2000).

Another problem encountered in every-day practice refers to the modelling of rigid joints or, more generally, of structural elements which can be assumed to behave as rigid bodies. Especially in the design and analysis of reinforced concrete foundations, massive footings are usually modelled by conventional beam elements with very large values as regards their moments of inertia. In order to simulate the elastic soil under the footings, these are modelled by a number of rigid beam elements supported elastically at their nodes by discrete Winkler springs or, alternatively, by a rigid beam supported at its middle by a translational and a rotational elastic spring. These simple techniques may yield acceptable results in the case of the Winkler soil. However, if a two-parameter soil model is employed, their application proves rather impractical. On the other hand, if axial force effects are neglected, exact solutions for Bernoulli/Timoshenko beam elements with rigid offsets at their ends and two-parameter elastic support throughout their length are available (Morfidis and Avramidis 2002).
An additional problem encountered in the design and analysis of steel structures relates to the modelling of flexible joint connections. A first approximation to this problem is the use of beam elements with rotational elastic springs at their ends (Matheu and Suarez 1996; Aristizabal – Ochoa 1997). Moreover, if the rigidity of the joints has to be taken into account, then a finite beam-column element with rigid offsets is indispensable. The connection between the rigid offsets and the median segment of the element is settled by rotational springs of appropriate stiffness. Such semi-rigid connections may be used for elastically supported beam-columns.
The objective of this paper is to exhaustively address all topics described above by means of a generalized finite beam-column element. This element is based on the exact analytical solution of the differential equation which describes the problem of an axially loaded member resting on an two-parameter elastic foundation, while featuring rigid offsets at its ends. The connection of rigid offsets to the interior element is achieved by means of rotational springs. The stiffness matrix is formed in a general way, which permits the use of either the Bernoulli or the Timoshenko member. The proposed new element is considered as generalized because of its ability to degenerate to various simpler elements: it is possible to separately ignore the rigid offsets (left or right or both), the semi-rigid connections (left or right or both) and even the elastic support. This is accomplished by zeroing certain coefficients in the expressions of the stiffness matrix, or by forming their limit values. These abilities render the generalized element very useful in structural analysis computer programs where, with the aid of appropriate "switches", it is possible to produce the desired element each time. In addition to the stiffness matrix, equivalent element nodal load vectors for a trapezoidal load with adjustable form parameters and for a linear temperature variation are given. The usefulness of the new beam-column element in the modeling and analysis of reinforced concrete and steel structures is documented by two numerical examples and comparisons to other less sophisticated solutions.

STIFFNESS MATRIX DERIVATION

The new generalized beam element is shown in Fig. 1a. It consists of two rigid segments between nodes 1,2 and 3,4 respectively (rigid offsets) and the flexible median segment between nodes 2 and 3, which is a Timoshenko beam element. For the connection of the median segment to the rigid offsets rotational springs are used. The element rests on a two-parameter elastic foundation and is loaded by a static axial force. When laying out the differential equations of the element, its deformed shape is taken into account.

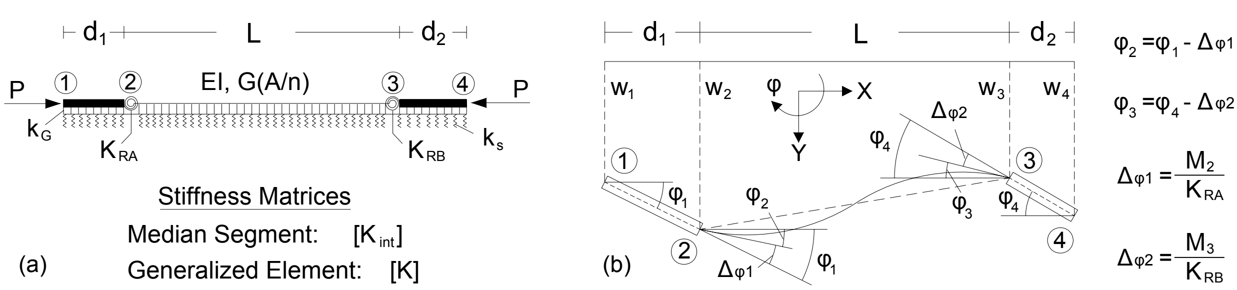


Fig. 1 (a) New generalized beam-column element, (b) Deformed configuration of the element

The derivation of the stiffness matrix [K] is accomplished in two stages:

1st. Formulation of the median segment's stiffness matrix [K_{int}], based on the analytical solution of its differential equations.

2nd. Formulation of the relations between the coefficients of the stiffness matrix [K_{int}] of the median segment 2-3 and those of the stiffness matrix [K] of the generalized element 1-4.

First stage

The exact stiffness matrix of the axially loaded Timoshenko beam on a two-parameter elastic foundation is derived by means of the analytical solution of the following differential equations:

$$EI\left[1-\frac{P-k_G}{\Phi}\right]\frac{d^4w}{dx^4}+\left[\left(P-k_G\right)-\frac{EIk_S}{\Phi}\right]\frac{d^2w}{dx^2}+k_S w+\frac{EI}{\Phi}\frac{d^2q}{dx^2}-q=0 \tag{1a}$$

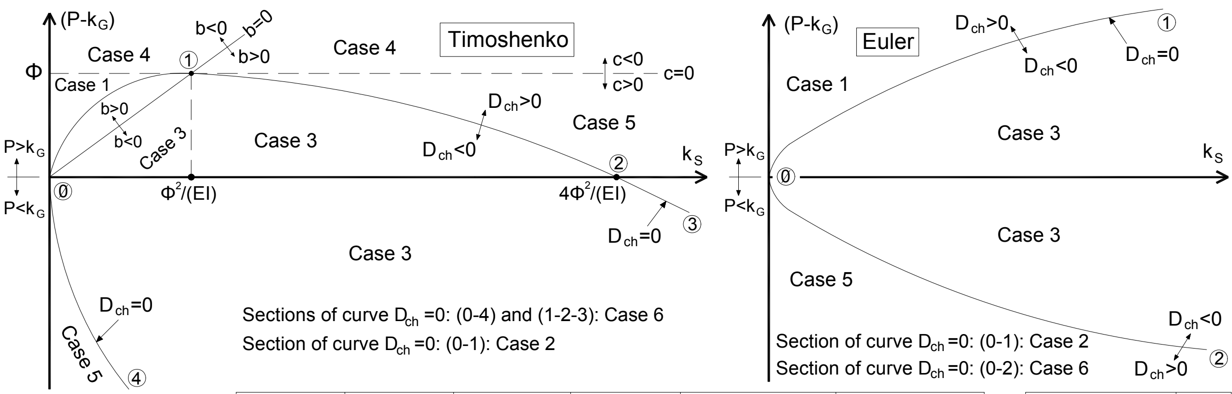
$$EI\left[1-\frac{P-k_G}{\Phi}\right]\frac{d^4\varphi}{dx^4}+\left[\left(P-k_G\right)-\frac{EIk_S}{\Phi}\right]\frac{d^2\varphi}{dx^2}+k_S\varphi-\frac{dq}{dx}=0 \tag{1b}$$

where EI is the flexural stiffness of the beam, Φ=G(A/n), φ is the cross-section's rotation due to flexure, A is the cross-section area, G is the shear modulus of elasticity, n is the shear factor, P is the axial load, q is the lateral external load, w is the lateral deflection, k_S (kN/m^2) is the modulus of the subgrade reaction, and k_G (kN) is the second parameter of the elastic foundation. The general forms of the analytical solutions of the equations mentioned above are:

$$w(x)=\sum_{i=1}^{4}C_i f_i(x)+w_p(x), \qquad \varphi(x)=\sum_{i=1}^{4}C'_i f_i(x)+\varphi_p(x) \tag{2}$$

The first terms of Eqs. (2) are the solutions of the homogenous form of equations (1a), (1b), while w_p, φ_p are the particular solutions corresponding to the external load q(x).

The two homogenous equations of (1a) and (1b) have the same structure, and possess six different forms of solution depending on the values of the coefficients describing the beam and soil properties (Fig. 2). However, it has been shown (Morfidis 2003, Avramidis and Morfidis 2004), that only case 1 and case 3 are of practical interest with regard to analyses according to second order theory (e.g., calculation of critical loads). The procedure for the analytical formulation of the median segment's stiffness matrix [K_{int}] for a Timoshenko beam on Winkler foundation has been presented by Cheng and Pantelides (1988). The extension of this procedure to Timoshenko beams on two-parameter elastic foundation, i.e., to Eqs. (1a) and (1b), and the corresponding coefficients of [K_{int}] for the two typical solution cases 1 and 3) have been presented by Morfidis (2003).



	b	c	D_{ch}	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	R	Q
Case 1	b>0	c>0	$D_{ch}>0$	cos(Rx)	sin(Rx)	cos(Qx)	sin(Qx)	$\sqrt{(1/2)(b-\sqrt{D_{ch}})}$	$\sqrt{(1/2)(b+\sqrt{D_{ch}})}$
Case 2	b>0	c>0	$D_{ch}=0$	cos(Qx)	xcos(Qx)	sin(Qx)	xsin(Qx)	-	$\sqrt{b/2}$
Case 3	$b\lessgtr 0$	c>0	$D_{ch}<0$	$e^{Rx}\cos(Qx)$	$e^{Rx}\sin(Qx)$	$e^{-Rx}\cos(Qx)$	$e^{-Rx}\sin(Qx)$	$\sqrt{\sqrt{(c/4)}-(b/4)}$	$\sqrt{\sqrt{(c/4)}+(b/4)}$
Case 4	$b\lessgtr 0$	c<0	$D_{ch}>0$	e^{Rx}	e^{-Rx}	cos(Qx)	sin(Qx)	$\sqrt{(1/2)(-b+\sqrt{D_{ch}})}$	$\sqrt{(1/2)(b+\sqrt{D_{ch}})}$
Case 5	b<0	c>0	$D_{ch}>0$	e^{Rx}	e^{-Rx}	e^{Qx}	e^{-Qx}	$\sqrt{(1/2)(-b+\sqrt{D_{ch}})}$	$\sqrt{(1/2)(-b-\sqrt{D_{ch}})}$
Case 6	b<0	c>0	$D_{ch}=0$	e^{Rx}	xe^{Rx}	e^{-Rx}	xe^{-Rx}	$\sqrt{-b/2}$	-

	Timoshenko	Euler
b	$\frac{1}{EI\theta}\left[(P-k_G)-\frac{EIk_S}{\Phi}\right]$	$\frac{P-k_G}{EI}$
c	$\frac{k_S}{EI\theta}$, $\theta=1-\frac{P-k_G}{\Phi}$	$\frac{k_S}{EI}$
D_{ch}	b^2-4c	

Fig. 2 Possible solution forms of equations (1a) and (1b) for Timoshenko and Euler beam elements

In case of Euler beams, the governing equations follow from Eqs. (1a) and (1b) by setting $\Phi \rightarrow \infty$. The cross-section's rotation due to flexure is now $\varphi \equiv dw/dx$, and, as a consequence, there is no coupling between Eqs. (1a) and (1b). In fact, Eq. (1b) follows from Eq. (1a) by a simple differentiation. Thus, a Euler beam on elastic two-parameter foundation is fully described by Eq. (1a) with $\Phi \rightarrow \infty$. Again, only cases 1 and 3 are of practical interest. The stiffness matrices $[K_{int}]$ for these cases are supplied by Karamanlidis and Prakash 1989.

Second stage

In order to formulate the relation between the stiffness matrix $[K_{int}]$ of the median segment and the stiffness matrix [K] of the generalized element, the following procedure is adopted:

a. Relations between the displacements of the internal nodes (2, 3), and the end nodes (1, 4), respectively.

On the basis of figure 1b, these relations are expressed as follows:

$$\begin{bmatrix} \varphi_2 \\ w_2 \\ \varphi_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d_1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -d_2 & 1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ w_1 \\ \varphi_4 \\ w_4 \end{bmatrix} + \begin{bmatrix} -(K_{RA})^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(K_{RB})^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} M_2 \\ V_2 \\ M_3 \\ V_3 \end{bmatrix} \tag{3}$$

where M_2 and M_3 are the bending moments at nodes 2 and 3 respectively, V_2 and V_3 are the shear forces, and K_{RA} and K_{RB} are the flexural stiffness of the rotational springs. Eq. (3) can be expressed in symbolic matrix form as:

$$[u_{\text{int}}] = [T][u] + [T_{KR}]\,[S_{\text{int}}] \tag{4}$$

b. Relations between the stresses at internal nodes 2,3 and nodes 1,4 respectively.

These relations are derived from the equilibrium conditions for the rigid offsets' free-body diagram (Fig. 3):

$$\begin{bmatrix} M_1 \\ V_{G1} \\ M_4 \\ V_{G4} \end{bmatrix} = \begin{bmatrix} 1 & d_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_2 \\ V_{G2} \\ M_3 \\ V_{G3} \end{bmatrix} + \left\{ \begin{bmatrix} \frac{1}{3}K_S b_{f1} d_1^3 & \frac{1}{2}K_S b_{f1} d_1^2 & 0 & 0 \\ \frac{1}{2}K_S b_{f1} d_1^2 & K_S b_{f1} d_1 & 0 & 0 \\ 0 & 0 & \frac{1}{3}K_S b_{f2} d_2^3 & -\frac{1}{2}K_S b_{f2} d_2^2 \\ 0 & 0 & -\frac{1}{2}K_S b_{f2} d_2^2 & K_S b_{f2} d_2 \end{bmatrix} + (P - k_G) \begin{bmatrix} -d_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\} \begin{bmatrix} \varphi_1 \\ w_1 \\ \varphi_4 \\ w_4 \end{bmatrix} \tag{5}$$

where K_S is the coefficient of subgrade reaction, and b_{f1}, b_{f2} are the width of the left and the right footing respectively. V_G is the "generalized shear force" (Vlasov and Leontiev 1966), which takes into account the effects of shearing stresses in the soil medium as well as the shearing stresses in the beam.

The coefficient of subgrade reaction K_S with dimension kN/m^3 must be distinguished from the modulus of subgrade reaction k_S with dimension kN/m^2. The relationship between K_S and k_S is given by $k_S = K_S b_B$, where b_B is the width of the foundation beam's cross-section. The terms $k_G d_1$ and $k_G d_2$ in (5) are moments resulting from the assumptions on which the two-parameter elastic foundation model is based, and are necessary for fulfilling the equilibrium conditions (Morfidis and Avramidis 2002). Equation (5) can be expressed in symbolic matrix form as:

$$[S] = [T]^T [S_{\text{int}}] + \{[K_W] + (P - k_G)[T_P]\}[u] \tag{6}$$

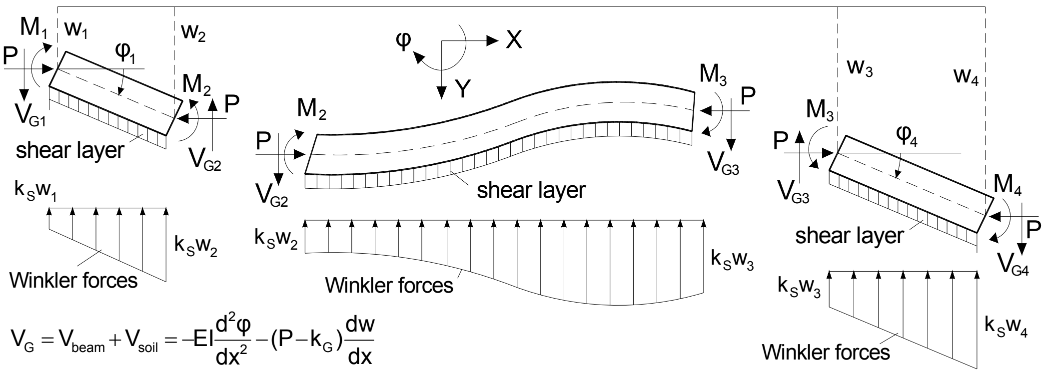


Fig. 3. Relationships between the forces at the internal joints and the element end forces

It is worth noticing that since rigid offsets cannot be deformed, the underlying shear layer remains inactive and does not transmit any force to them. Consequently, the only forces transmitted to the rigid offsets are the vertical spring forces, as in the case of the Winkler model.

c. Formulation of the stiffness matrix [K] as a function of the stiffness matrix [K_{int}].

In this step, the classic matrix equation stating the force-deformation relationship is considered:

– For the median segment: $[S_{\text{int}}] = [K_{\text{int}}]\,[u_{\text{int}}]$ (7)

where [K_{int}] is the already known stiffness matrix of the median segment 2-3.

– For the whole element: $[S] = [K]\,[u]$ (8)

where [K] is the stiffness matrix of the generalized element 1-4 yet to be derived.

From Eqs. (4) and (7) we obtain:

$$[S_{\text{int}}] = [K_{\text{int}}]\,\{[T]\,[u] + [T_{KR}]\,[S_{\text{int}}]\,\} \tag{9}$$

After some algebra, Eq. (9) gives:

$$[S_{\text{int}}] = \{[I] - [K_{\text{int}}]\,[T_{KR}]\}^{-1}[K_{\text{int}}]\,[T]\,[u] \tag{10}$$

where [I] is the 4x4 identity matrix. By combining Eqs. (6) and (10) we obtain:

$$[S] = [T]^T\{[I] - [K_{\text{int}}]\,[T_{KR}]\}^{-1}[K_{\text{int}}]\,[T]\,[u] + \{\,[K_W] + (P - k_G)[T_P]\,\}\,[u]\,. \tag{11}$$

Finally, the comparison of Eqs. (8) (11) leads to f the stiffness matrix of the generalized element:

$$[K] = [T]^T\{[I] - m[K_{\text{int}}]\,[T_{KR}]\}^{-1}[K_{\text{int}}]\,[T]\; + n[K_W]\; + (P - nk_G)[T_P] \tag{12}$$

In Eq. (12) certain "switches" have been incorporated in order to enable the optional omitting of either the elastic support or the rotational springs. Moreover, a "switch" can convert the geometric nonlinear stiffness matrix to a geometric linear one (Morfidis and Avramidis 2002). A "switch" is a parameter that takes the unity value if a certain effect must be taken into account, or the zero value if this effect must be neglected. If m=0, then Eq. (12) states the geometric nonlinear stiffness matrix of a beam element without semi-rigid connections, supported by one or two-parameter elastic foundation. If n=0, then Eq. (12) expresses the geometric nonlinear stiffness matrix of a beam element without elastic support, which is applicable in the analysis of steel structures. If m=P=0 and n=1, then Eq. (12) states the linear stiffness matrix of a beam element with one or two-parameter elastic support, which is applicable in the analysis of reinforced concrete foundations. If k_G=P=0 and n=1, then Eq. (12) states the linear stiffness matrix of a beam element with one-parameter elastic support. Finally, if m=n=P=0, then Eq. (12) represents the well-known linear stiffness matrix of a beam element with rigid offsets (Ghali and Neville 1989).

ELEMENT NODAL LOAD VECTORS

In this paragraph, the equivalent nodal load vectors for a trapezoidal load with adjustable form parameters and for a linear temperature variation Δt between top and bottom fibers of the beam are formed (Fig. 4).

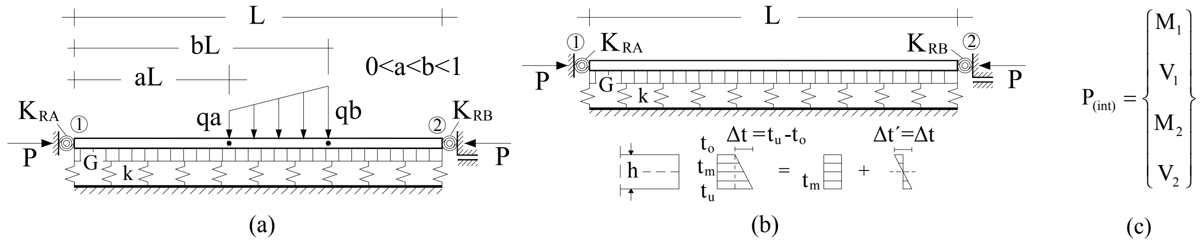


Fig. 4 (a) Transverse trapezoidal load, (b) linear temperature variation Δt, (c) general form of internal load vector

The load vectors (as well as the stiffness matrices) are based on the exact solution of the governing differential equations of the problem and are derived in two steps. First, the load vectors [$P_{(int)}$] for the median segment of the element with rotational springs at its ends are formed. Then, the stresses are transmitted through the rigid offsets to the end nodes of the whole element to form the equivalent element load vectors. For both load cases, the load vectors for the Bernoulli beam element result from the load vectors of the respective Timoshenko beam element by forming the limit value of the latter, as shear rigidity Φ=AG/n approaches infinity.

Results for load vectors for the cases 1 and 3 are included in Appendix 1 and Appendix 2.

Trapezoidal load with adjustable form parameters

The nodal load vector of the Timoshenko beam element supported by a two-parameter elastic foundation is derived by means of the analytical solution of Eqs. (1a) and (1b) and by applying the method of initial parameters (e.g. Morfidis, 2003).

Consideration of the rigid offsets:

In order to derive the load vector of the new element, the relationships between the stresses at the auxiliary nodes 2 and 3, and the stresses at the nodes 1 and 4 must be formulated. These relationships are derived from the equilibrium conditions associated to the free-body diagram of the rigid offsets. Assuming that their load is uniform and equal to q_{ro} one takes:

$$\begin{bmatrix} M_1 \\ V_{G1} \\ M_4 \\ V_{G4} \end{bmatrix} = \begin{bmatrix} 1 & d_1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} M_2 \\ V_{G2} \\ M_3 \\ V_{G3} \end{bmatrix} + (q_{ro}) \begin{bmatrix} -d_1^2 \\ -2d_1 \\ d_2^2 \\ -2d_2 \end{bmatrix} \quad \Rightarrow \quad [P^{(tr)}] = [T]^T [P^{(tr)}_{(\text{int})}] + (q_{ro})[T_{tr}] \tag{13}$$

Non-uniform temperature variation Δt

In this case, the load vector results from Eqs. (1a) and (1b), in which the value of q is set equal to 0 (Morfidis, Avramidis 2002). Moreover, the relationship from which the bending moments are determined is:

$$M(x) = -EI(\frac{d\varphi}{dx} + \frac{\alpha \Delta t}{h}) \quad (\alpha \text{ is the coefficient of thermal expansion})$$

Consideration of the rigid offsets:

The consideration of the rigid offsets leads to the following equation:

$$[P^{(\Delta t)}] = [T]^T [P^{(\Delta t)}_{(\text{int})}] \tag{14}$$

EXAMPLES

The two examples presented below aim to test the reliability of the new element in modeling of plane systems, as well as to indicate the simplification and reduction of computational sources achieved with its use. In order to solve these examples, computer programs in Fortran 90/95 that include the new element were created while, for the evaluation of the reliability of the results, comparative solutions were attempted using the well known structural analysis program SAP2000 Nonlinear Version 7.42 (2000).

Example 1

The first example concerns the steel frame of Fig. 5(a) that rests on the elastic soil through a reinforced concrete foundation (foundation beam and footings). The soil consists of a layer of loose sand 20m in thickness, resting on a rigid base. Young's modulus and Poisson's ratio of the sand are E_S=17500kN/m^2 and ν_S=0.28 respectively (Bowles 1988). With regard to given frame, two categories of analyses were carried out:

A) The objective of the first category of analyses was the investigation of the reliability of the new element in modeling of beams on elastic foundation. In a first step, the two soil parameters are determined by an analysis using the modified Vlasov soil model, proposed by Vallabhan and Das (Vallabhan and Das 1991). Their values were found to be k_S=3321.3kN/m^2 and k_G=11143.19kN (assuming plane stress conditions). Next, a number of analyses are carried out using the Winkler model, in which the same value of k_S is used as in the case of the modified Vlasov model, i.e. k_S=3321.3kN/m^2. Finally, an additional analysis using the Pasternak model is performed, which was based on soil constants resulting from an analysis using the modified Vlasov model.

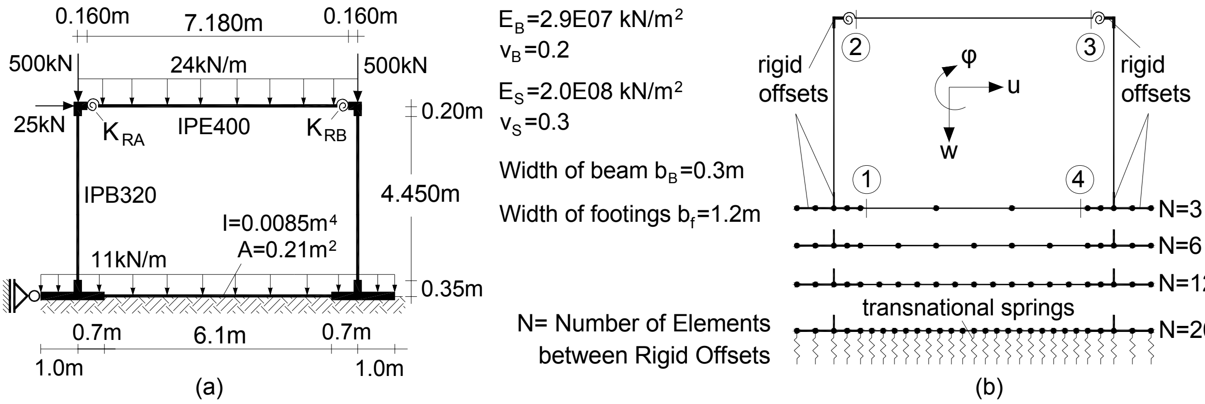


Fig. 5 (a) Steel frame with R/C foundation beam and footings on elastic soil, (b) Four conventional discretization models of increasing mesh density

Solutions based on the Winkler model were achieved both using the new element, and using four different conventional models of the foundation beam (Fig. 5b). The main objective of the Winkler-model-analyses was to prove that the new element can, in fact, yield results that may only be attained using a large number of simple (classical) beam elements in case of conventional modeling. In addition, conventional modeling of continuous support requires additional calculations in order to determine the stiffnesses of the discrete springs required.

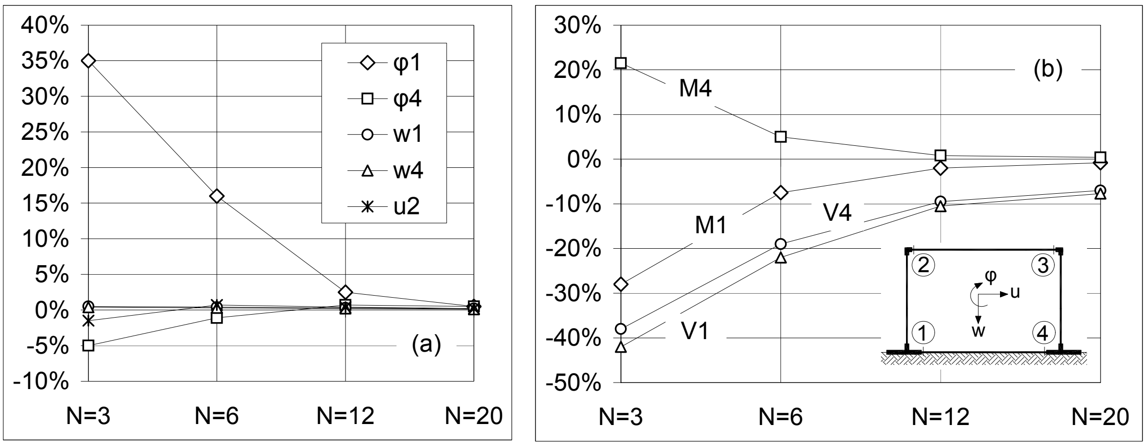


Fig 6. Deviations of nodal displacements (a) and stresses (b) of the four conventional models relative to the reference solution based on the proposed generalized element

A careful study of Fig. 6 leads to the following conclusions:

- The model with N=3 simple elements fails to approximate the “exact” values of the stresses, as it displays deviations of the order of 25% for bending moments and 40% for shear forces.
- The model with N=6 simple elements, although better in comparison with the previous one, still exhibits considerable deviations, especially for the shear forces (20%). However, divergences are relatively small when it comes to bending moments (below 10%).
- The model with N=12 simple elements gives far better results in comparison with the previous one, the deviations remaining below 15%.
- The model with N=20 elements consists of the minimum number of elements necessary for an absolutely satisfactory approximation of all stress values. In this particular case, the deviations do not exceed 7%.

The analyses in which the two-parameter models were employed aimed at investigating the influence of the second parameter on the stress values of the foundation beam. Apart from that, in the case of the solution based on the Vlasov model, the influence of the soil on either side of the frame was also taken into account by placing translational springs with constants $\sqrt{k_S k_G}$ (Vlasov and Leontiev 1966) at the ends of the foundation beam (nodes 5 and 6, Fig. 7a). Figure 7a indicates that the difference between the Winkler and Pasternak models with reference to the maximum bending moment of the beam is of the order of 20%, whereas the respective difference between the Winkler and Vlasov models approaches 3400%! From figure 7b, it is evident that the difference between the Winkler and Pasternak models in terms of shear forces does not exceed 10% at any given point of the beam. The respective differences between the Winkler and Vlasov models reach at certain points 80-90%. Therefore, the consideration of the soil on either side of the frame substantially modifies its stress condition, since it alters the way in which it rests on the soil.

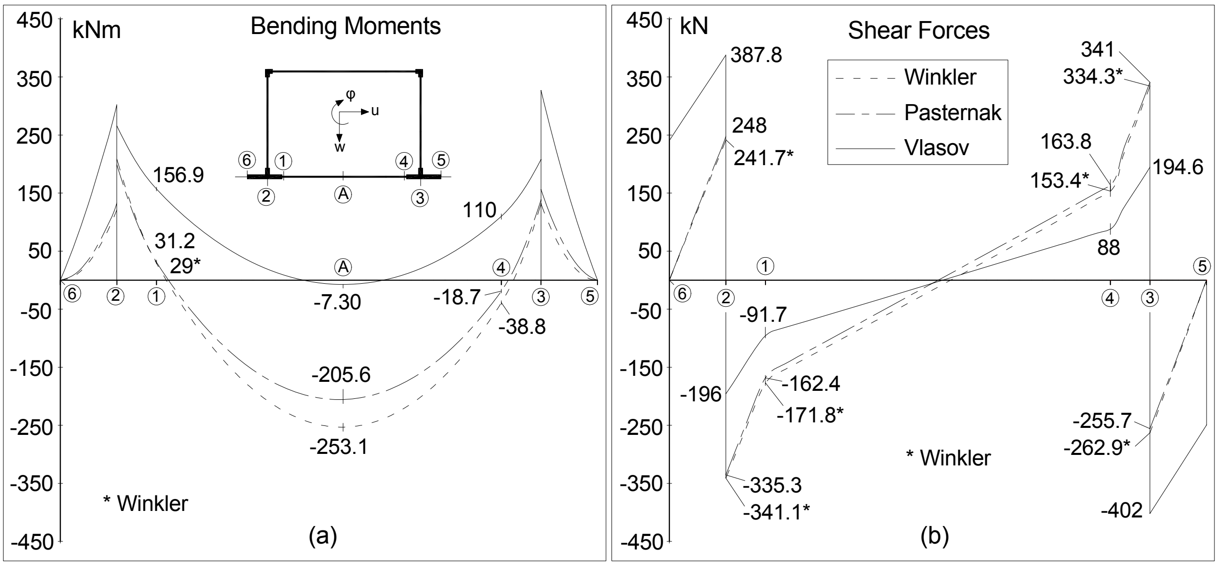


Fig. 7 Diagrams of bending moments and shear forces in the foundation beam for Winkler, Pasternak and Vlasov models

B) The objective of the second category of analyses was to provide evidence for the usefulness and practicality of the new element in modeling of semi-rigid connections in steel structures. In the conventional solution using the well-known structural analysis program SAP2000, the modeling of semi-rigid connections was based on two different connection models (Fig. 8a).

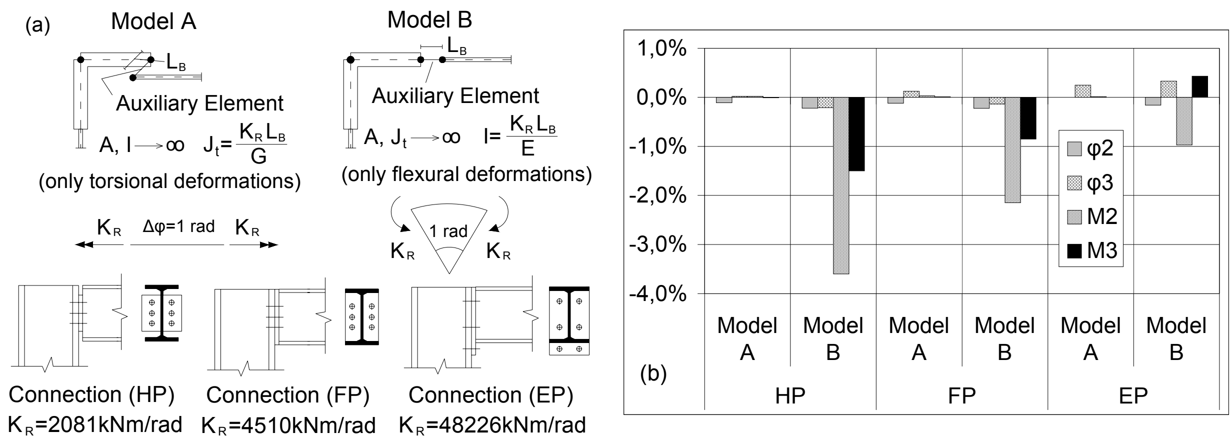


Fig. 8 (a) Conventional modeling of semi-rigid connections using additional auxiliary elements, (b) Deviations of the conventional model analyses from the analyses based on the proposed generalized element

The solutions were carried out using three different values for the rotational stiffness of the semi-rigid connections (Council on tall buildings and urban habitat, 1993). The comparisons of rotations and bending moments at the semi-rigid connections are illustrated in figures 8a and 8b. From the these diagrams, it is concluded that the conventional modeling of semi-rigid connections (Fig. 8a) leads to values of stresses and displacements that are very close to those derived from the analysis using the new element. However, this does not diminish the usefulness of the new element since the deviation of results requires: (a) extensive preparatory work in order to determine the appropriate lengths of the auxiliary elements used, as well as the geometric properties of their sections (Fig. 8a), and (b) the introduction of additional auxiliary nodes and elements in the model.

It is clear, that a more complicated frame, with more bays and stories, would demand multiple repetitions of this conventional modeling procedure, thus dramatically increasing pre- and postprocessing efforts. In contrast, using the proposed new element simplifies the modeling of steel structures with semi-rigid connections to a remarkably large extent.

Example 2

The second example concerns the buckling load evaluation of a foundation structure composed of a foundation beam rigidly connected to three piles which lower ends rest on a hard bedrock (Fig. 9a). The surrounding soil is considered to behave elastic in both vertical and horizontal directions. Here, it is assumed that the superstructure (consisting of three shear walls) is very stiff and that the possibility of instability is limited to the foundation structure. The piles transmit their vertical load to the rigid bedrock entirely through their lower end point and not through axial friction along the shaft. The modeling of reactions due to friction (e.g, using axially oriented springs along the pile's axis) is thereby rendered redundant. With these assumptions, the sufficiently realistic model shown in figure 9b can be used, while the minimum number of conventional finite elements necessary for numerically acceptable results is still to be specified. Here, five different variations as to the discretization's density are investigated (Fig. 9b).

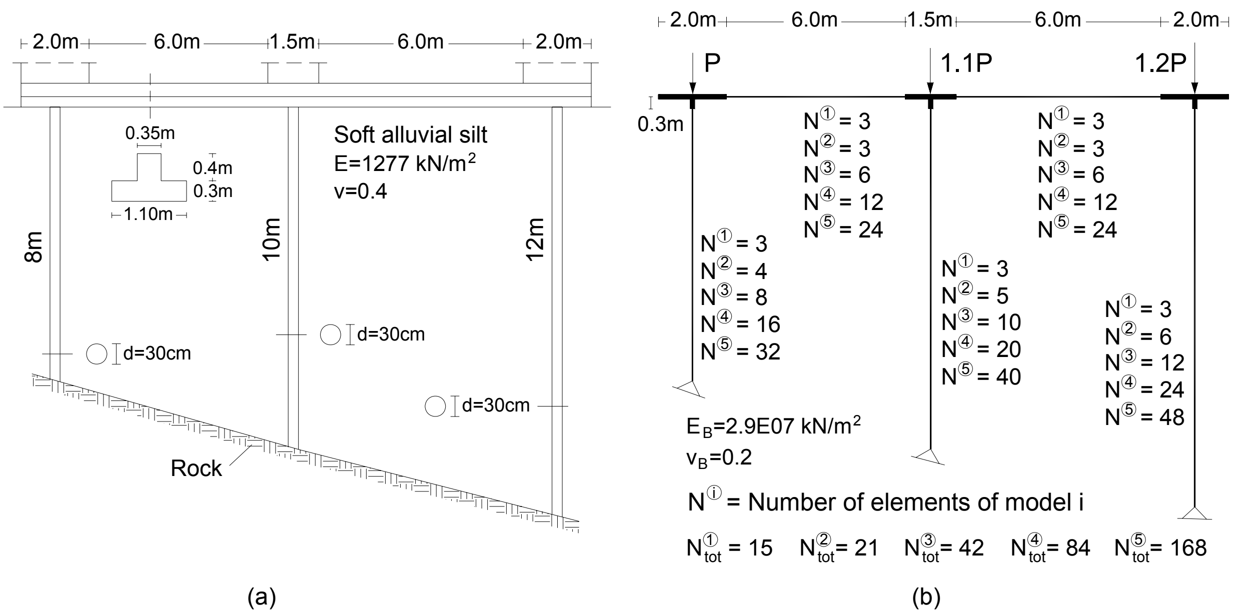


Fig.9. Foundation structure (beam and piles), (b) Five conventional discretization models of increasing mesh density

Solutions were given using both the Winkler model and the two-parameter model. Concerning the numerical values of the first parameter, i.e., the modulus of vertical subgrade reaction k_S for the foundation beam and the modulus of horizontal subgrade reaction k_h for the piles, the following assumptions were made:

a. The foundation soil consists of soft alluvial silt. Therefore, the modulus of horizontal subgrade reaction k_h may be considered as constant with depth (Terzaghi, 1955). The elastic constants used are E_S=1277kN/m^2 and ν_S=0.4 (Bowles, 1988).

b. For the moduli k_S and k_h the analytical relation proposed by Vesic (Vesic, 1961) is used:

$$K_S = \left[0.65\left(\sqrt[12]{\frac{E_S \bullet b_B^4}{EI}}\right) \bullet \frac{E_S}{b_B(1-\nu_S^2)}\right] kN/m^3 \Rightarrow k_S = (K_S \bullet b_B) kN/m^2 \tag{15}$$

Thus:

$$\text{Vertical Subgrade Reaction}: K_S = \left(\sqrt[12]{\frac{1277 \bullet 1.1^4}{(2.9 \bullet 10^7) \bullet 0.0164}}\right) \bullet \left(\frac{0.65 \bullet 1277}{1.1 \bullet (1-0.4^2)}\right) = 566.2 kN/m^3 \Rightarrow k_S = 622.8 kN/m^2$$

$$\text{Horizontal Subgrade Reaction}: K_h = \left(\sqrt[12]{\frac{1277 \bullet 0.3^4}{(2.9 \bullet 10^7) \bullet (3.98 \bullet 10^{-4})}}\right) \bullet \left(\frac{0.65 \bullet 1277}{0.3 \bullet (1-0.4^2)}\right) = 1835.5 kN/m^3 \Rightarrow k_h = 550.7 kN/m^2$$

c. The numerical analyses were caried out for two different values of K_h: (I) K_h=K_S (according to Bowles, 1988), and (II) K_h=2K_S (according toTerzaghi, 1955).

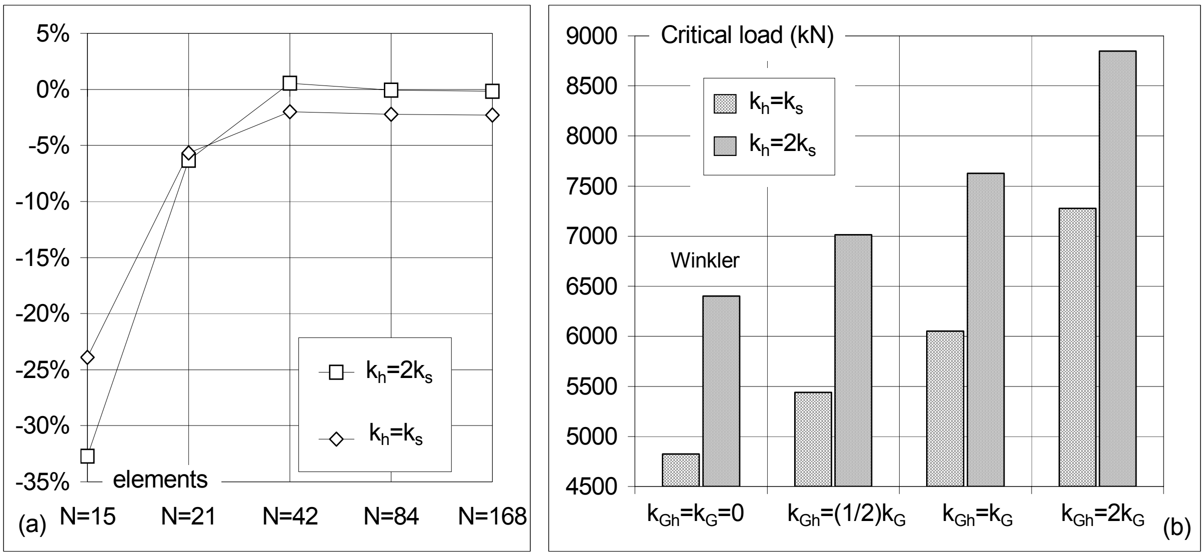
Thus: (I) k_S=622.8kN/m^2, k_h=550.7kN/m^2, and (II) k_S=622.8kN/m^2, k_h=1101.4kN/m^2

In order to calculate the value of the second parameter, i.e., k_G for the foundation beam and k_{Gh} for the piles, the Vallabhan and Das model (Vallabhan and Das 1991) was employed. On the basis of the corresponding procedure, with E_S=1277kN/m^2, ν_S=0.4 and an average depth of the silt layer equal to 10m, the value k_G=1457.2 kN for the second parameter of the elastic foundation comes up. This value is used for the foundation beam, while for the piles, in order to investigate the influence of the second parameter's value on the buckling load, three different values k_{Gh}=(1/2)k_G, k_{Gh}=k_G, and k_{Gh}=2k_G are used. Thus:

(I) k_G=1457.2kN → k_{Gh}=728.6kN, (II) k_G=1457.2kN → k_{Gh}=1457.2kN, and (III) k_G=1457.2kN → k_{Gh}=2914.4kN.

Various series of numerical calculations based on the above assumptions have been carried out. The outcome can be summarized as follows:

A) A first set of results refers to results produced by the "exact" solution, i.e., the one using the proposed element, as compared to the corresponding Winkler model results based on conventional analyses of different different discretization densities (Fig. 9b). These comparisons were made for both cases: $k_h=k_S$ and $k_h=2k_S$. Figure 10a



$$\text{Deviation}(\%) = \left[\frac{(\text{CMV})}{(\text{NMV})} - 1\right] \cdot 100$$

$$(\text{CMV}) = (\text{Conventional Model Value})$$
$$(\text{RMV}) = (\text{New Model Value - Model with new element})$$

Fig.10. (a) Deviations of the conventional finite element model buckling loads from the buckling load based on the proposed generalized element, (b) Buckling loads for different combinations of soil parameter values

depicts the divergences of the conventional model buckling load values from buckling load values provided by analyses using the proposed element. As shown in this figure, the approximation of the exact value of the buckling load with a divergence lower than 1% is achieved through the use of conventional models made up of at least 42 conventional beam elements. Models consisting of 21 elements yield results with divergences of the order of 7%, which may be deemed as acceptable. Finally, models comprising only 15 elements display divergences of the order of 30% and should, therefore, be rejected.

B) In a second set of analyses, the new element was exclusively employed, while the elastic subgrade was simulated using (a) the Winkler model, and (b) the two-parameter model. These analyses aimed at specifying the fluctuation of the value of the critical buckling load, depending on the assumptions made as to the the values of the soil parameter k_{Gh}. Fig. 10b leads to the following conclusions:

1. As far as solutions based on the Winkler model are concerned, the difference between the value of the critical load in case $k_h=2k_S$, and its value in case of $k_h=k_S$ is of the order of 35%.
2. The use of the two-parameter model generally increases the values of the critical load. The assumption concerning the correlation between k_{Gh} and k_G is of significant influence. More specifically:

- If $k_h=k_S$, the change in the value of the critical load between the assumptions $k_{Gh}=(1/2)k_G$ and $k_{Gh}=2k_G$ is of the order of 35%.
- If $k_h=2k_S$, the change in the value of the critical load between the assumptions $k_{Gh}=(1/2)k_G$ and $k_{Gh}=2k_G$ is of the order of 25%.

3. The difference between the two-parameter model and the Winkler model ranges from 14% to 53%, if $k_h=k_S$, and from 9% to 38%, if $k_h=2k_S$.

SYNOPSIS AND CONCLUSIONS

In the present paper, a new generalized Bernouli/Timoshenko beam-column element is developed, which proves particularly usefull in modelling R/C or steel structural members in many cases arising in every-day practice. The element consists of a flexible median segment, which is a classical Euler or Timoshenko beam, and of two rigid segments on both sides of it. Median segment and rigid offsets are connected by rotational elastic springs and supported by a two-parameter elastic foundation. The final form of the generalized element stiffness matrix as well as of the corresponding load vectors contain a number of appropriate "switches" which allow for the optional activation or deactivation of one or more of the above element characteristics. Thus, the proposed element can be adapted to a variety of specific situations and modelling needs. The derivation of its stiffness matrix and load vectors is based on the exact analytical solution of the differential equations which describe the problem of the axially loaded generalized element resting on an two-parameter elastic foundation. Among the six cases of solution in case of Timoshenko beam and the five cases of solution in case of Euler beam only two cases proved to be of practical interest. For these cases, the corresponding matrices and vectors are developed.

The efficiency of the proposed generalized beam-column element has been demonstrated by two numerical examples. In both examples axial force effects have been taken into account.

A main conclusion drawn from comparisons with analyses using conventional finite beam elements is that the use of generalized element minimizes the number of finite elements necessary for achieving acceptable results. This is due to the ability of the generalized element to incorporate both rigid offsets and rotational spring connections without needing additional nodes, i.e., without causing additional degress of freedom.

A second conclusion refers to the favorable ability of the two-parameter soil models to simulate the influence of the foundation soil on either side of foundation beams. One-parameter models lack this ability, thus leeding to significant deviations from the correct stresses in the foundation structure. The presented generalized element, in its version with active two-parameter elastic foundation, is capable of handling with this problem even in presence of axial load effects.

Finally, a general conclusion from buckling load investigations of elastically embedded structures is that two-parameter soil models generally leed to greater values for the buckling load as compared to one-parameter models. The buckling load increases not only with increasing value of the modulus of horizontal subgrade reaction k_h, which is rather expected, but also with increasing value of the second soil parameter (k_G for beams resting on elastic soil, and k_{Gh} for beams or piles embedded in elastic soil). Thus, as far as buckling loads are concerned, the two-parameter soil models proved less conservative than the one-parameter models. Further investigations are needed in order to determine whether the two-parameter models possibly overstimate the buckling loads of elastically supported or/and elastically embedded structures.

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APPENDIX 1: Load vector coefficients for non-uniform temperature variation Δt (see Figures 2 and 4)

Case 1

$$M_1 = \frac{(EI)^2\alpha\Delta t}{hD_{\Delta t}}\left[\Delta_1(R+Q)-\Delta_2(R-Q)\right]\left\{\frac{EI}{K_{RA}K_{RB}}\left(v^2-\mu^2\right)\left[\Delta_1(R+Q)-\Delta_2(R-Q)\right]-v\left(\Delta_1-\Delta_2\right)\left[\frac{v'}{K_{RA}}+\frac{\mu'}{K_{RB}}\right]+\mu\left(\Delta_1+\Delta_2\right)\left[\frac{\mu'}{K_{RA}}+\frac{v'}{K_{RB}}\right]\right\}$$

$$+\left(EI\alpha\Delta t/h\right)$$

$$V_1 = \frac{(EI)^2\alpha\Delta t}{hD_{\Delta t}}\left\{\Delta_2(R-Q)^2\left[\frac{F_1}{K_{RA}}-\frac{F_2}{K_{RB}}\right]+\Delta_1(R+Q)^2\left[\frac{F_3}{K_{RA}}+\frac{F_4}{K_{RB}}\right]\right\}-\frac{EI(P-k_G)\alpha\Delta t}{hD_{\Delta t}}\left\{(R-Q)\left[\frac{F_1}{K_{RA}}-\frac{F_2}{K_{RB}}\right]+(R+Q)\left[\frac{F_3}{K_{RA}}+\frac{F_4}{K_{RB}}\right]\right\}$$

where:

$$F_1 = \frac{1}{2}\left(\frac{EI}{K_{RB}}\right)\left(\bar{v}+\bar{\mu}\right)\left[\left(\Delta_1-\Delta_2\right)R+\left(\Delta_1+\Delta_2\right)Q\right]+\left(\Delta_1+\Delta_2\right)\mu^2+\left(\Delta_1-\Delta_2\right)v^2 \qquad F_2 = \frac{EI}{K_{RA}}\left[\left(\Delta_1-\Delta_2\right)R+\left(\Delta_1+\Delta_2\right)Q\right]\left(v\mu'+\mu v'\right)+2\Delta_1\mu v$$

$$F_3 = -\frac{1}{2}\left(\frac{EI}{K_{RB}}\right)\left(\bar{v}-\bar{\mu}\right)\left[\left(\Delta_1-\Delta_2\right)R+\left(\Delta_1+\Delta_2\right)Q\right]+\left(\Delta_1+\Delta_2\right)\mu^2-\left(\Delta_1-\Delta_2\right)v^2 \qquad F_4 = \frac{EI}{K_{RA}}\left[\left(\Delta_1-\Delta_2\right)R+\left(\Delta_1+\Delta_2\right)Q\right]\left(v\mu'-\mu v'\right)+2\Delta_2\mu v$$

$$D_{\Delta t} = -\frac{(EI)^2}{K_{RA}K_{RB}}\left(v^2-\mu^2\right)\left[\left(\Delta_1-\Delta_2\right)R+\left(\Delta_1+\Delta_2\right)Q\right]^2+\frac{EI}{2}\left(\frac{1}{K_{RA}}+\frac{1}{K_{RB}}\right)\left[\left(\Delta_1-\Delta_2\right)R+\left(\Delta_1+\Delta_2\right)Q\right]\left[\left(\Delta_1-\Delta_2\right)\bar{v}-\left(\Delta_1+\Delta_2\right)\bar{\mu}\right]+$$

$$+\left[\left(\Delta_1-\Delta_2\right)^2v^2-\left(\Delta_1+\Delta_2\right)^2\mu^2\right]$$

$$\Delta_{1/2} = \frac{1}{R\pm Q}\left[\theta\left(R\pm Q\right)^2+\frac{k}{\Phi}\right] \qquad \Phi=GF' \qquad v=sin(RL) \qquad v'=cos(RL) \qquad \mu=sin(QL)$$

$$\theta = 1-\left[\left(P-k_G\right)/\Phi\right] \qquad \mu'=cos(QL) \qquad \bar{v}=sin(2RL) \qquad \bar{\mu}=sin(2QL)$$

Case 3

$$M_1 = \frac{(EI)^2\alpha\Delta t}{hD_{\Delta t}}\left(\omega_2R+\omega_1Q\right)\left\{\frac{2EI}{K_{RA}K_{RB}}\left(\omega_2R+\omega_1Q\right)\left[\left((n')^2-n^2\right)-\left((m')^2+m^2\right)\right]+4\omega_1n\left(\frac{n'}{K_{RA}}+\frac{m'}{K_{RB}}\right)-4\omega_2m\left(\frac{m'}{K_{RA}}+\frac{n'}{K_{RB}}\right)\right\}+\frac{EI\alpha\Delta t}{h}$$

$$V_1 = -\frac{4EI\alpha\Delta t}{hD_{\Delta t}}\left\{\frac{EI}{K_{RA}K_{RB}}\left(\omega_2R+\omega_1Q\right)\left(n'-m'\right)\left[(EI)F_1+\left(P-k_G\right)\left(nR+mQ\right)\right]+\left[(P-k_G)F_2+(EI)F_3\right]\right\}$$

where:

$$F_1 = \left(R^2-Q^2\right)\left(n\omega_1+m\omega_2\right)+2RQ\left(m\omega_1-n\omega_2\right) \qquad F_2 = R\left[\left(\frac{\omega_1}{K_{RA}}\right)n^2-\left(\frac{\omega_2}{K_{RB}}\right)mn\right]-Q\left[\left(\frac{\omega_2}{K_{RA}}\right)m^2-\left(\frac{\omega_1}{K_{RB}}\right)mn\right]$$

$$F_3 = \frac{1}{K_{RA}}\left(R^2-Q^2\right)\left(\omega_1^2n-\omega_2^2m\right)-2RQ\left\{\omega_2n\left[\left(\frac{\omega_1n}{K_{RA}}\right)-\left(\frac{\omega_2m}{K_{RB}}\right)\right]+\omega_1m\left[\left(\frac{\omega_2m}{K_{RA}}\right)-\left(\frac{\omega_1n}{K_{RB}}\right)\right]\right\}$$

$$D_{\Delta t} = \frac{(2EI)^2}{K_{RA}K_{RB}}\left(\omega_2R+\omega_1Q\right)^2\left(m^2+n^2\right)+4EI\left[\frac{1}{K_{RA}}+\frac{1}{K_{RB}}\right]\left(\omega_2R+\omega_1Q\right)\left(\omega_2m\,m'-\omega_1n\,n'\right)+4\left(\omega_2^2m^2-\omega_1^2n^2\right)$$

$$\omega_1 = \frac{RA_1+QA_2}{R^2+Q^2} \qquad \omega_2 = \frac{-QA_1+RA_2}{R^2+Q^2} \qquad A_1 = \theta(R^2-Q^2)-(k/\Phi) \qquad A_2 = 2RQ\theta$$

$n=sin(QL)$ $\qquad$ $n'=cos(QL)$ $\qquad$ $m=sinh(RL)$ $\qquad$ $m'=cosh(RL)$

Formulae for R and Q are given in Fig.2.

The expressions for M_2 and V_2 follow from the above expressions for M_1 and V_1 by mutual interchange of the rotation springs' coefficients K_{RA} and K_{RB}.

APPENDIX 2: Load vector coefficients for trapezoidal load (see Figures 2 and 4)

$$M_1 = \frac{1}{D_Q}\left[\frac{1}{K_{RA}}\left(F_1 I_w - F_3 I_M\right) + \left(I_G F_3 - I_w F_2\right)\right]$$

$$V_1 = \frac{1}{D_Q}\left[\frac{1}{K_{RA}K_{RB}}\left(F_5 I_w - F_1 I_M\right) + \frac{1}{K_{RA}}\left(F_1 I_G - F_4 I_w\right) + \frac{1}{K_{RB}}\left(F_2 I_M - F_4 I_w\right) + \left(F_6 I_w - F_2 I_G\right)\right]$$

$$where: D_Q = \frac{1}{K_{RA}K_{RB}}\left(F_5 F_3 - F_1^2\right) + \left(\frac{1}{K_{RA}} + \frac{1}{K_{RB}}\right)\left(F_1 F_2 - F_3 F_4\right) + \left(F_3 F_6 - F_2^2\right)$$

The expressions for M_2 and V_2 follow from the above expressions for M_1 and V_1 by mutual interchange of the rotation springs' coefficients K_{RA} and K_{RB}.

Case 1

$$F_1 = (g_t - d_t)v\mu' + (g_t + d_t)v'\mu \qquad F_2 = a_t v\mu \qquad F_3 = (c_t - f_t)v\mu' - (c_t + f_t)v'\mu$$

$$F_4 = v'\mu' - (g_t\Delta_1 + d_t\Delta_2)v\mu \qquad F_5 = EI\left[(g_t\Delta_1 Z_1 - d_t\Delta_2 Z_2)v\mu' + (g_t\Delta_1 Z_1 + d_t\Delta_2 Z_2)\mu v'\right]$$

$$F_6 = a_t\left[(\Delta_1 - \Delta_2)v\mu' + (\Delta_1 + \Delta_2)v'\mu\right] \qquad (Z_1 = R + Q \qquad Z_2 = R - Q)$$

$$I_w = \bar{q}_a\left[(c_t - f_t)I_{f3} - (c_t + f_t)I_{f4}\right] + \frac{(q_b - q_a)}{L(b-a)}\left[(c_t - f_t)I_{g3} - (c_t + f_t)I_{g4}\right] \qquad I_G = 2a_t\left[\bar{q}_a I_{f2} + \frac{(q_b - q_a)}{L(b-a)}I_{g2}\right]$$

$$I_V = \bar{q}_a\left[I_{f1} - (b_t + e_t)I_{f2}\right] + \frac{(q_b - q_a)}{L(b-a)}\left[I_{g1} - (b_t + e_t)I_{g2}\right] \qquad \bar{q}_a = q_a - \left[\frac{q_b - q_a}{L(b-a)}\right](aL)$$

$$I_M = \bar{q}_a\left[(g_t - d_t)I_{f3} + (g_t + d_t)I_{f4}\right] + \frac{(q_b - q_a)}{L(b-a)}\left[(g_t - d_t)I_{g3} + (g_t + d_t)I_{g4}\right]$$

$$I_{f1} = \frac{(R+Q)(v'_a - v'_b) - (R-Q)(v_a - v_b)}{2(R^2 - Q^2)} \qquad I_{f2} = \frac{(R+Q)(v'_a - v'_b) + (R-Q)(v_a - v_b)}{2(R^2 - Q^2)}$$

$$I_{f3} = -\frac{(R+Q)(\mu'_a - \mu'_b) + (R-Q)(\mu_a - \mu_b)}{2(R^2 - Q^2)} \qquad I_{f4} = \frac{(R+Q)(\mu'_a - \mu'_b) - (R-Q)(\mu_a - \mu_b)}{2(R^2 - Q^2)}$$

$$I_{g1} = \frac{-1}{2(R^2 - Q^2)^2}\left\{(R+Q)^2(\mu'_a - \mu'_b) + (R-Q)\left[(R-Q)(\mu_a - \mu_b) + (R+Q)^2(-aLv'_a + bLv'_b) + (R^2 - Q^2)(aLv_a - bLv_b)\right]\right\}$$

$$I_{g2} = \frac{-1}{2(R^2 - Q^2)^2}\left\{(R+Q)^2(\mu'_a - \mu'_b) - (R-Q)\left[(R-Q)(\mu_a - \mu_b) + (R+Q)^2(aLv'_a - bLv'_b) + (R^2 - Q^2)(aLv_a - bLv_b)\right]\right\}$$

$$I_{g3} = \frac{1}{2(R^2 - Q^2)^2}\left\{-(R^2 - Q^2)\left[(R+Q)(aL\mu'_a - bL\mu'_b) + (R-Q)(aL\mu_a - bL\mu_b)\right] - (R+Q)^2(v'_a - v'_b) + (R-Q)^2(v_a - v_b)\right\}$$

$$I_{g4} = \frac{1}{2(R^2 - Q^2)^2}\left\{(R^2 - Q^2)\left[(R+Q)(aL\mu'_a - bL\mu'_b) - (R-Q)(aL\mu_a - bL\mu_b)\right] + (R+Q)^2(v'_a - v'_b) + (R-Q)^2(v_a - v_b)\right\}$$

$$v_a = \sin\left[L(a-1)(R+Q)\right] \quad v'_a = \sin\left[L(a-1)(Q-R)\right] \quad \mu_a = \cos\left[L(a-1)(R+Q)\right] \quad \mu'_a = \cos\left[L(a-1)(Q-R)\right]$$

$$v_b = \sin\left[L(b-1)(R+Q)\right] \quad v'_b = \sin\left[L(b-1)(Q-R)\right] \quad \mu_b = \cos\left[L(b-1)(R+Q)\right] \quad \mu'_b = \cos\left[L(b-1)(Q-R)\right]$$

$$v = \sin(RL) \quad v' = \cos(RL) \quad \mu = \sin(QL) \quad \mu' = \cos(QL)$$

$$a_t = D_R^{-1}\left[EI\Delta_1\Delta_2\left(Z_1^2 - Z_2^2\right) + (P - k_G)(\Delta_1 Z_2 - \Delta_2 Z_1)\right] \quad b_t = (EI\Delta_2 Z_2)a_t \quad c_t = D_R^{-1}EI\Delta_1(\Delta_1 Z_1 - \Delta_2 Z_2)$$

$$d_t = D_R^{-1}EIZ_1(\Delta_1 Z_1 - \Delta_2 Z_2)\left[EI\Delta_1 Z_1 - (P - k_G)\right] \quad e_t = (EI\Delta_1 Z_1)a_t \quad f_t = D_R^{-1}\left[EI\Delta_2(\Delta_1 Z_1 - \Delta_2 Z_2)\right]$$

$$g_t = D_R^{-1}EIZ_2(\Delta_1 Z_1 - \Delta_2 Z_2)\left[EI\Delta_2 Z_2 - (P - k_G)\right]$$

$$D_R = -EI(\Delta_1 Z_1 - \Delta_2 Z_2)\left[EI\Delta_1\Delta_2\left(Z_1^2 - Z_2^2\right) + (P - k_G)(\Delta_1 Z_2 - \Delta_2 Z_1)\right]$$

Case 3

$$F_1 = (2b_t)mn' - (2f_t)m'n \quad F_3 = (2a_t)mn' - (2d_t)m'n \quad F_5 = -2EI[(A_1 b_t - A_2 f_t)mn' - (A_2 b_t + A_1 f_t)m'n]$$

$$F_2 = (2c_t)mn \quad F_4 = m'n' - 2(\omega_2 b_t + \omega_1 f_t)mn \quad F_6 = 2c_t(\omega_2 mn' + \omega_1 m'n)$$

$$I_w = 2\overline{q}_a(a_t I_{f3} - d_t I_{f4}) + \frac{2(q_b - q_a)}{L(b-a)}(a_t I_{g3} - d_t I_{g4}) \quad I_M = 2\overline{q}_a(b_t I_{f3} - f_t I_{f4}) + \frac{2(q_b - q_a)}{L(b-a)}(b_t I_{g3} - f_t I_{g4})$$

$$I_G = 2c_t\left[\overline{q}_a I_{f2} + \frac{(q_b - q_a)}{L(b-a)} I_{g2}\right] \quad I_V = \overline{q}_a[I_{f1} + (2e_t)I_{f2}] + \frac{(q_b - q_a)}{L(b-a)}[I_{g1} + (2e_t)I_{g2}] \quad \overline{q}_a = q_a - \left[\frac{q_b - q_a}{L(b-a)}\right](aL)$$

$$I_{f1} = (R^2 + Q^2)^{-1}[Q(m_b\overline{n}_b - m_a\overline{n}_a) + R(\overline{m}_b n_b - \overline{m}_a n_a)] \quad I_{f3} = (R^2 + Q^2)^{-1}[R(\overline{m}_a m_a - \overline{m}_b m_b) + Q(\overline{n}_a n_a - \overline{n}_b n_b)]$$

$$I_{f2} = (R^2 + Q^2)^{-1}[Q(\overline{m}_a n_a - \overline{m}_b n_b) + R(\overline{m}_b n_b - m_a\overline{n}_a)] \quad I_{f4} = (R^2 + Q^2)^{-1}[R(\overline{n}_a n_a - \overline{n}_b n_b) + Q(\overline{m}_b m_b - \overline{m}_a m_a)]$$

$$I_{g1} = (R^2 + Q^2)^{-2}\{(R^2 + Q^2)[bL(R\overline{m}_b n_b + Qm_b\overline{n}_b) - aL(R\overline{m}_a n_a + Qm_a\overline{n}_a)] + (R^2 - Q^2)(\overline{m}_a m_a - \overline{m}_b m_b) + 2RQ(\overline{n}_a n_a - \overline{n}_b n_b)\}$$

$$I_{g2} = (R^2 + Q^2)^{-2}\{(R^2 + Q^2)[aL(Q\overline{m}_a n_a - Rm_a\overline{n}_a) + bL(Rm_b\overline{n}_b - Q\overline{m}_b n_b)] + (R^2 - Q^2)(\overline{n}_a n_a - \overline{n}_b n_b) - 2RQ(\overline{m}_a m_a - \overline{m}_b m_b)\}$$

$$I_{g3} = (R^2 + Q^2)^{-2}\{(R^2 + Q^2)[aL(R\overline{m}_a m_a + Qn_a\overline{n}_a) - bL(Rm_b\overline{m}_b + Q\overline{n}_b n_b)] + (R^2 - Q^2)(\overline{m}_b n_b - \overline{m}_a n_a) + 2RQ(m_b\overline{n}_b - m_a\overline{n}_a)\}$$

$$I_{g4} = (R^2 + Q^2)^{-2}\{(R^2 + Q^2)[aL(Rn_a\overline{n}_a - Q\overline{m}_a m_a) + bL(Qm_b\overline{m}_b - R\overline{n}_b n_b)] + (R^2 - Q^2)(m_b\overline{n}_b - m_a\overline{n}_a) - 2RQ(\overline{m}_b n_b - \overline{m}_a n_a)\}$$

$$n_a = \sinh[RL(a-1)] \quad m_a = \cosh[RL(a-1)] \quad \overline{n}_a = \sin[QL(a-1)] \quad \overline{m}_a = \cos[QL(a-1)] \quad n = \sin(QL) \quad m = \sinh(RL)$$

$$n_b = \sinh[RL(b-1)] \quad m_b = \cosh[RL(b-1)] \quad \overline{n}_b = \sin[QL(b-1)] \quad \overline{m}_b = \cos[QL(b-1)] \quad n' = \cos(QL) \quad m' = \cosh(RL)$$

$$a_t = D_R^{-1}[2EI(\omega_1 Q + \omega_2 R)\omega_2] \quad b_t = D_R^{-1}\{2EI(\omega_1 Q + \omega_2 R)[EI(2RQ\omega_1 + (R^2 - Q^2)\omega_2) + (P - k_G)Q]\}$$

$$c_t = -1/[2EI(\omega_1 Q + \omega_2 R)] \quad d_t = D_R^{-1}[2EI\omega_1(R\omega_2 + Q\omega_1)]$$

$$e_t = (1/2)[(\omega_2 Q - \omega_1 R)/(\omega_1 Q + \omega_2 R)] \quad f_t = D_R^{-1}\{2EI(\omega_1 Q + \omega_2 R)[(P - k_G)R + EI((R^2 - Q^2)\omega_1 - 2RQ\omega_2)]\}$$

$$D_R = 4EI(\omega_1 Q + \omega_2 R)[2EIRQ(\omega_1^2 + \omega_2^2) + (P - k_G)(\omega_1 Q - \omega_2 R)]$$

Formulae for R and Q are given in Fig.2.

Parameters Δ_1, Δ_2, ω_1, ω_2, A_1 and A_2 are the same as in case of non-uniform temperature variation Δt (see Appendix 1).